

**APPLICATION OF QUINTIC SPLINE COLLOCATION METHOD FOR TWO
DIMENSIONAL LAMINAR FLOW BETWEEN TWO MOVING POROUS
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Abstract: In this paper, a steady two dimensional flow of a viscous incompressible fluid in a rectangular domain that is bounded by two permeable surfaces. The Quintic Spline Collocation method has been implemented to obtain a solution of the fourth order nonlinear boundary value problem. It was found that for all values of Parameters the Suggested method agrees well with available result. Numerical and Graphical result obtained show excellent agreement with earlier results represented in the literature.

Key words: Two dimensional flow, Viscous incompressible fluid, Quintic Spline Collocation, fourth order nonlinear boundary value problem.

1. INTRODUCTION

Studies of fluid transport in biological organisms often concern the flow of a particular fluid inside an expanding or contracting vessel with permeable walls. Permeable wall is important to the mass transfer between blood, air and tissue^[1]. Research work has been invested in the study of the flow in rectangular bounded by two moving porous walls, which enable the fluid to enter or exit during successive expansion or contractions. Dauenhauer and Majdalani^[3] studied the unsteady flow in semi-infinite expanding channels with wall injections; they are characterized by two non dimensional parameters, the expansion ratio of the wall α and the cross-flow Reynolds number. Majdalani and Zou^[4] studied moderate to large injection and suction driven channel flows with expanding or contracting walls.

Spline collocation method was introduced by Dr.H. Doctor, Jigisha pandya^[5,9]. In this study Quintic spline collocation method (QSC) applied to approximate solutions of nonlinear differential equations governing Two-dimensional viscous flow through expanding or contracting gaps with permeable walls and have comparison with Homotopy Analysis method (HMA), Numerical Method, Spectral homotopy analysis method (SHAM).

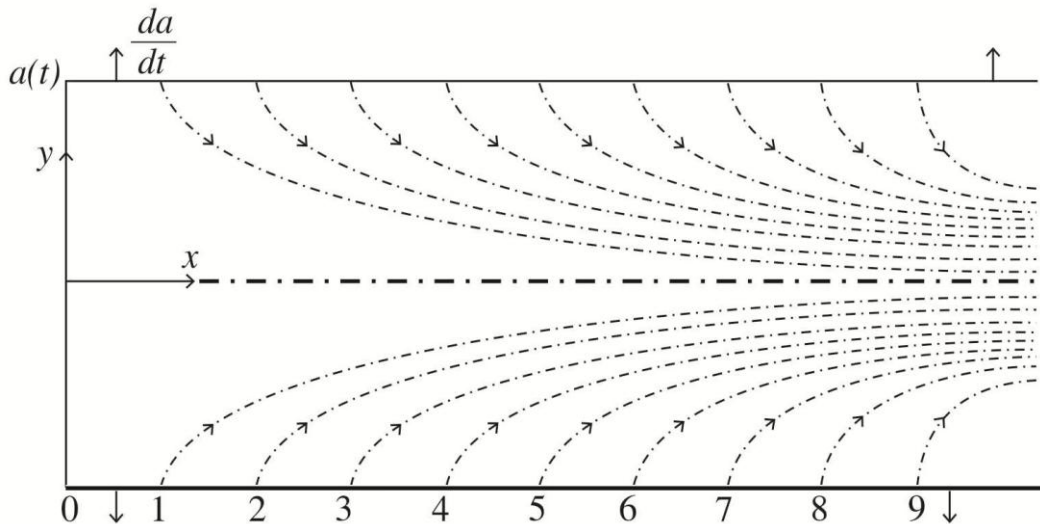
2. FORMULATION OF THE PROBLEM

Consider two-dimensional laminar, isothermal, and incompressible viscous fluid flow in a rectangular domain bounded by two permeable surfaces that enable the fluid to enter or exit during expansions or contractions. The walls are placed at a separation $2a$ and contract or expand uniformly at a time-dependent rate $\dot{a}(t)$. The governing Navier-Stokes equations are given in Majdalani and Zodwa et al.[2,6] as

$$\begin{aligned}\frac{\partial \hat{u}}{\partial \hat{x}} + \frac{\partial \hat{v}}{\partial \hat{y}} &= 0, \\ \frac{\partial \hat{u}}{\partial t} + \hat{u} \frac{\partial \hat{u}}{\partial \hat{x}} + \hat{v} \frac{\partial \hat{u}}{\partial \hat{y}} &= -\frac{1}{\rho} \frac{\partial \hat{p}}{\partial \hat{x}} + \nu \nabla^2 \hat{u}, \\ \frac{\partial \hat{v}}{\partial t} + \hat{u} \frac{\partial \hat{v}}{\partial \hat{x}} + \hat{v} \frac{\partial \hat{v}}{\partial \hat{y}} &= -\frac{1}{\rho} \frac{\partial \hat{p}}{\partial \hat{y}} + \nu \nabla^2 \hat{v},\end{aligned}\tag{1)-(2}$$

Where $\hat{u}$ and $\hat{v}$ are the velocity components in the $\hat{x}$ and $\hat{y}$ directions, respectively, $\hat{p}$, ρ , ν and t are the dimensional pressure, density, kinematic viscosity, and time, respectively. Assuming that inflow or outflow velocity is v_w , Then the boundary conditions are

(3)



Where $c = \left(\frac{\dot{a}}{v_w} \right)$ is the injection or suction coefficient. Introducing the stream function $\psi = \sqrt{x} \hat{F}(y, t) / a$ and the transformations

$$\psi = \frac{\hat{\psi}}{a \hat{a}}, \quad u = \frac{\hat{u}}{\hat{a}}, \quad v = \frac{\hat{v}}{\hat{a}}, \quad x = \frac{\hat{x}}{a}, \quad y = \frac{\hat{y}}{a}, \quad F = \frac{\hat{F}}{\text{Re}},$$

Majdalani et al. [2] and Dinarvand and Rashidi [7] showed that (1)-(2) reduced to the normalized nonlinear differential equation

$$F^{IV} + \alpha(yF''' + 3F'') + \text{Re}(FF''' - F'F'') = 0 \quad (4)$$

Subject to the boundary conditions

$$\begin{aligned} F = 0, \quad F'' = 0, \quad & \text{at } y = 0, \\ F = 1, \quad F' = 0, \quad & \text{at } y = 1, \end{aligned} \quad (5)$$

Where $\alpha(t) = \dot{a}a/v$ is the no dimensional wall dilation rate defined to be positive for expansion and negative for contraction, and $\text{Re} = av_w/v$ is the filtration Reynolds number defined positive for injection and negative for suction through the walls. Equation (4) is strongly nonlinear and not easy to solve analytically, and most researchers have studied the classic Berman formula [8]; that is when $\alpha = 0$, In this paper, we seek to solve (4) subject to the boundary conditions (5) using Quintic Spline Collocation method. This technique is more accurate and efficient with the comparison of the previously obtained result.

After the flow field is found the normal pressure gradient can be obtained by substituting the velocity components in to equations (1)-(2).

$$p_y = -[\text{Re}^{-1} f'' + ff' + \alpha \text{Re}^{-1}(f + yf')], \quad p = \frac{\hat{p}}{\rho V_w^2}. \quad (6)$$

We can determine the normal pressure distribution, if we integrate equation (6)

$$(7)$$

$$\Delta p_n = \text{Re}^{-1} f'(0) - [\text{Re}^{-1} f' + ff' + \alpha \text{Re}^{-1}(f + yf')]. \quad (8)$$

3. Quintic Spline Collocation Method

The fifth degree spline is used to find numerical solutions to the boundary value problems discussed in (4) together with (5). A detailed description of spline functions generated by subdivision is given by de Boor [10].

Consider equally spaced knots of a partition $\pi: a = x_0 < x_1 < x_2 < \dots < x_n = b$ on $[a, b]$. Let $S_5[\pi]$ be the space of continuously differentiable, piecewise, Quintic polynomials on π . That is, $S_5[\pi]$ is the space of Quintic polynomials on π . The Quintic spline is given by Bickley [11] and by G. Micula and S. Micula [12]

$$s(x) = a_0 + b_0(x - x_0) + \frac{1}{2}c_0(x - x_0)^2 + \frac{1}{6}d_0(x - x_0)^3 + \frac{1}{24}(x - x_0)^4 + \frac{1}{120} \sum_{k=0}^{n-1} F_k (x - x_k)_+^5 \quad (9)$$

where the power function $(x - x_k)_+$ is defined as

$$(x - x_k)_+ = \begin{cases} x - x_k, & \text{if } x > x_k, \\ 0, & \text{if } x \leq x_k, \end{cases} \quad (10)$$

Consider a fourth – order linear boundary value problem of the form

$$y^{iv}(x) + p(x)y'''(x) + q(x)y''(x) + r(x)y'(x) + t(x)y(x) = m(x), \quad a \leq x \leq b, \quad (11)$$

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Subject to the boundary conditions

$$\begin{aligned} \alpha_0 y_0 + \beta_0 y'_0 + \gamma_0 y''_0 + \delta_0 y'''_0 &= \eta_0, \\ \alpha_1 y'_0 + \beta_1 y_1 + \gamma_1 y''_1 + \delta_1 y'''_1 &= \eta_1, \\ \alpha_2 y''_0 + \beta_2 y_2 + \gamma_2 y''_2 + \delta_2 y'''_2 &= \eta_2, \\ \alpha_3 y'''_0 + \beta_3 y_3 + \gamma_3 y''_3 + \delta_3 y'''_3 &= \eta_3, \end{aligned} \quad (12)$$

Where $y(x), p(x), q(x), r(x), t(x)$ and $m(x)$ are continuous functions defined in the interval $x \in [a, b]$; $\eta_0, \eta_1, \eta_2, \eta_3$ are finite real constants.

Let (9) be an approximate solution of (11), where $a_0, b_0, c_0, d_0, e_0, F_0, F_1, \dots, F_{n-1}$ are real coefficients to be determined.

Let $x_0, x_1, \dots, x_n$ be $n+1$ grid points in the interval $[a, b]$, so that

$$x_i = a + ih, \quad i = 0, 1, \dots, n; \quad x_0 = a, \quad x_n = b, \quad h = \frac{b-a}{n}$$

It is required that the approximate solution (9) satisfies the differential equation at the point $x = x_i$. Putting (9) with its successive derivatives in (11), we obtain the collocation equations as follows:

$$\begin{aligned}
 & \sum_{k=0}^{n-1} F_k \left\{ (x_i - x_k)_+ + \frac{1}{2} p(x_i)(x_i - x_k)_+^2 + \frac{1}{6} q(x_i)(x_i - x_k)_+^3 + \frac{1}{24} r(x_i)(x_i - x_k)_+^4 + \frac{1}{120} t(x_i)(x_i - x_k)_+^5 \right\} \\
 & + e_0 \left\{ 1 + p(x_i)(x_i - x_0) + \frac{1}{2} q(x_i)(x_i - x_0)^2 + \frac{1}{6} r(x_i)(x_i - x_0)^3 + \frac{1}{24} t(x_i)(x_i - x_0)^4 \right\} \\
 & + d_0 \left\{ p(x_i) + q(x_i)(x_i - x_0) + \frac{1}{2} r(x_i)(x_i - x_0)^2 + \frac{1}{6} t(x_i)(x_i - x_0)^3 \right\} \\
 & + c_0 \left\{ q(x_i) + r(x_i)(x_i - x_0) + \frac{1}{2} t(x_i)(x_i - x_0)^2 \right\} + b_0 \{ r(x_i) + t(x_i)(x_i - x_0) \} + a_0 \{ t(x_i) \} \\
 & = m(x_i), \quad i = 0, 1, 2, \dots, n.
 \end{aligned} \tag{13}$$

From boundary conditions,

$$\begin{aligned}
 & \sum_{k=0}^{n-1} F_k \left(\frac{\delta_0}{2} (b - x_k)_+^2 + \frac{\gamma_0}{6} (b - x_k)_+^3 \right) + e_0 \left(\delta_0 (b - a) + \frac{\gamma_0}{2} (b - a)^2 \right) + d_0 (\delta_0 + \gamma_0 (b - a)) \\
 & + c_0 (\gamma_0) + b_0 (\beta_0) + a_0 (\alpha_0) = \eta_0, \\
 & \sum_{k=0}^{n-1} F_k \left(\frac{\delta_1}{2} (b - x_k)_+^2 + \frac{\gamma_1}{6} (b - x_k)_+^3 + \frac{\beta_1}{120} (b - x_k)_+^5 + \right) \\
 & + e_0 \left(\delta_1 (b - a) + \frac{\gamma_1}{2} (b - a)^2 + \frac{\beta_1}{24} (b - a)^4 \right) \\
 & + d_0 \left(\delta_1 + \gamma_1 (b - a) + \frac{\beta_1}{6} (b - a)^3 \right) + c_0 \left(\gamma_1 + \frac{\beta_1}{2} (b - a)^2 + \gamma_1 (b - a) + \alpha_1 \right) \\
 & + b_0 (\beta_1 (b - a) + \alpha_1) + a_0 (\beta_1) = \eta_1, \\
 & \sum_{k=0}^{n-1} F_k \left(\frac{\delta_2}{2} (b - x_k)_+^2 + \frac{\gamma_2}{6} (b - x_k)_+^3 + \frac{\beta_2}{120} (b - x_k)_+^5 + \right) \\
 & + e_0 \left(\delta_2 (b - a) + \frac{\gamma_2}{2} (b - a)^2 + \frac{\beta_2}{24} (b - a)^4 \right) \\
 & + d_0 \left(\delta_2 + \gamma_2 (b - a) + \frac{\beta_2}{6} (b - a)^3 \right) + c_0 \left(\gamma_2 + \frac{\beta_2}{2} (b - a)^2 + \gamma_2 (b - a) + \alpha_2 \right) \\
 & + b_0 (\beta_2 (b - a)) + a_0 (\beta_2) = \eta_2, \\
 & \sum_{k=0}^{n-1} F_k \left(\frac{\delta_3}{2} (b - x_k)_+^2 + \frac{\gamma_3}{6} (b - x_k)_+^3 + \frac{\beta_3}{120} (b - x_k)_+^5 + \right)
 \end{aligned}$$

$$\begin{aligned}
 & + e_0 \left(\delta_3 (b-a) + \frac{\gamma_3}{2} (b-a)^2 + \frac{\beta_3}{24} (b-a)^4 \right) \\
 & + d_0 \left(\delta_3 + \gamma_3 (b-a) + \frac{\beta_3}{6} (b-a)^3 + \alpha_3 \right) + c_0 \left(\gamma_3 + \frac{\beta_3}{2} (b-a)^2 \right) \\
 & + b_0 (\beta_3 (b-a)) + \alpha_0 (\beta_3) = \eta_3,
 \end{aligned} \tag{14}$$

Using the power function $(x-x_k)_+$ in the above equations, a system of $n+5$ linear equations in $n+5$ unknowns $a_0, b_0, c_0, d_0, e_0, F_0, F_1, \dots, F_{n-1}$ is thus obtained. This system can be written in matrix-vector form as follows:

$$AX = B, \tag{15}$$

Where $X = [F_{n-1}, F_{n-2}, \dots, F_2, F_1, F_0, e_0, d_0, c_0, b_0, a_0]^T$, $B = [\eta_3, \eta_2, \eta_1, \eta_0, m(x_n), m(x_{n-1}), \dots, m(x_1), m(x_0)]^T$.

The coefficient matrix A is an upper triangular Hessenberg matrix with a single lower sub diagonal, principal and upper diagonal having nonzero elements. Because of this nature of matrix A, the determination of the required quantities becomes simple and consumes less time. The values of these constants ultimately yield the quintic spline $s(x)$ in (9).

In case of nonlinear boundary value problem, the equations can be converted into linear form using quasilinearization method (Bellman and Kalaba [13]), and hence this method can be used as iterative method. The procedure to obtain a spline approximation of y_i ($i = 0, 1, 2, \dots, j$, where j denotes the number of iteration) by an iterative method starts with fitting a curve satisfying the end conditions and this curve is designated as y_i . We obtain the successive iterations y_i 's with the help of an algorithm described as above till desired accuracy.

Application of Quasilinearization and quintic spline method

$$F^{IV} + \alpha(yF''' + 3F'') + \text{Re}(FF''' - F'F'') = 0 \tag{16}$$

Boundary conditions

$$F = 0, \quad F'' = 0, \quad \text{at } y = 0, \tag{17}$$

Quasilinearization technique $F = 1, \quad F' = 0, \quad \text{at } y = 1,$

$$f_{i+1}^{iv} + (\text{Re } f_i) f_{i+1}''' - m^2 f_{i+1}'' + (\text{Re } f_{i+1}') f_{i+1}' = \text{Re } f_i''' f_i \tag{18}$$

Linearized boundary conditions

$$f_{i+1}(0) = 0, \quad f_{i+1}''(0) = 0, \tag{19}$$

The quintic spline

$$f_{i+1}(1) = 1, \quad f_{i+1}'(1) = 0, \tag{20}$$

Substitute (20) in (18) and (19)

(21) From first two boundary conditions

$$a_0 = 0, \quad c_0 = 0, \quad (22)$$

From other two boundary conditions

$$\frac{1}{24} \sum_{k=0}^{n-1} f_k (z_i - z_k)_+^4 + \frac{1}{6} e_0 (z_i - z_0)^3 + \frac{1}{2} d_0 (z_i - z_0)^2 + c_0 (z_i - z_0) + b_0 = 0$$

$$\frac{1}{120} \sum_{k=0}^{n-1} f_k (z_i - z_k)_+^5 + \frac{1}{24} e_0 (z_i - z_0)^4 + \frac{1}{6} d_0 (z_i - z_0)^3 + \frac{1}{2} c_0 (z_i - z_0)^2 + b_0 (z_i - z_0) + a_0 = 1$$

Solving the above system with initial spline approximation $s(z)$ of $F(z)$ described in equation (16) satisfying the boundary conditions (17), a curve $F(z) = \frac{-1}{2} z^3 + \frac{3}{2} z$ is assumed to be fitted through the points $z=0$ and $z=1$, we find the results, shown in graphs.

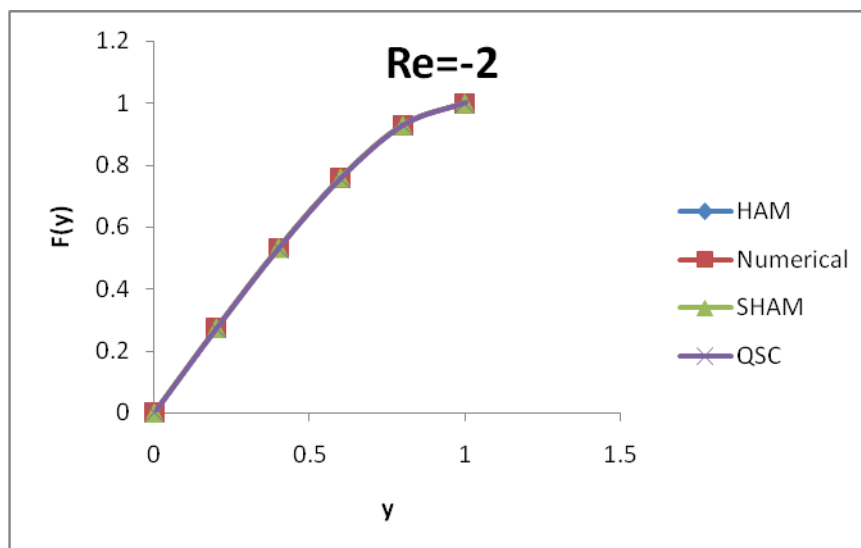


Figure:2

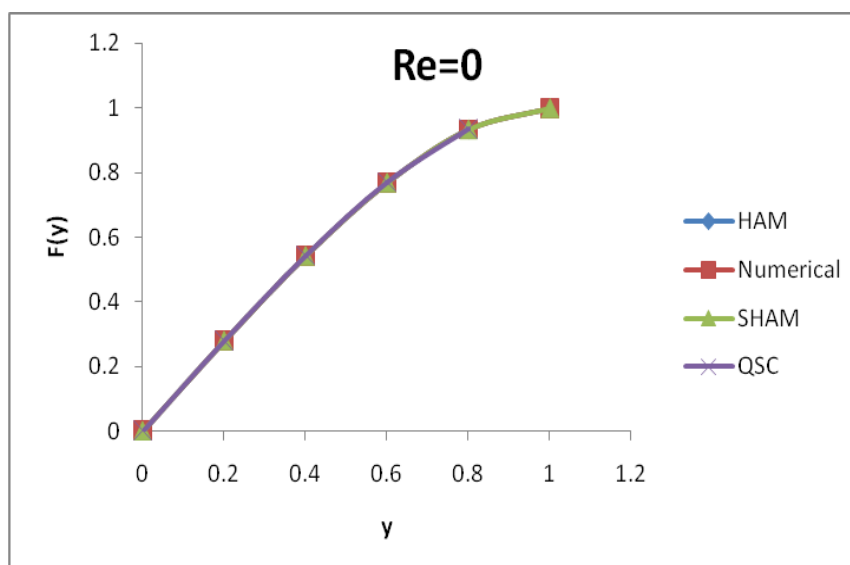


Figure:3

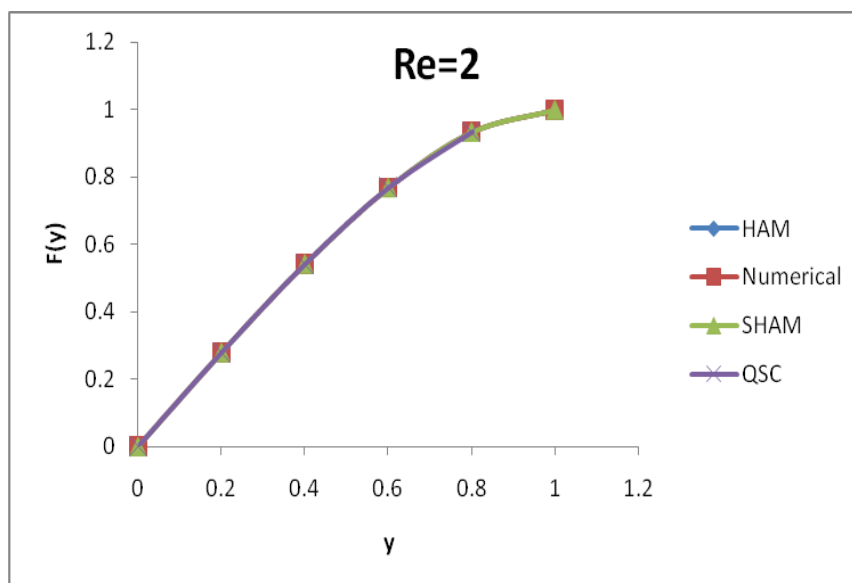
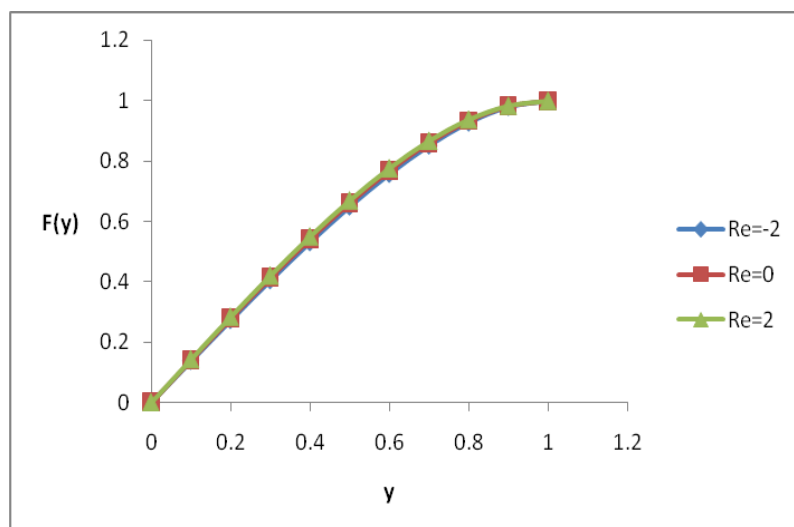


Figure:4

Figure 2,3,4 : Comparison between HAM, Numerical,SHAM and QSC for (y) for alpha=-1



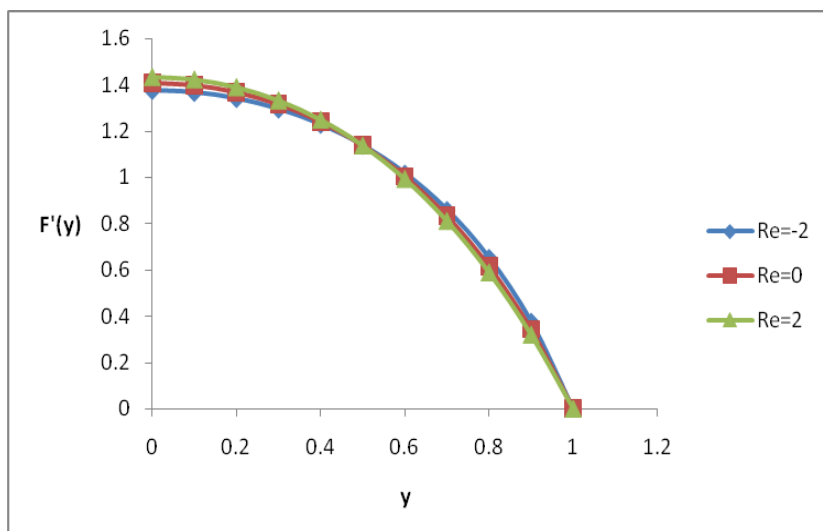


Figure 5: approximate solution of $F(y)$ and $F'(y)$ for different values of Re when $\alpha=-1$

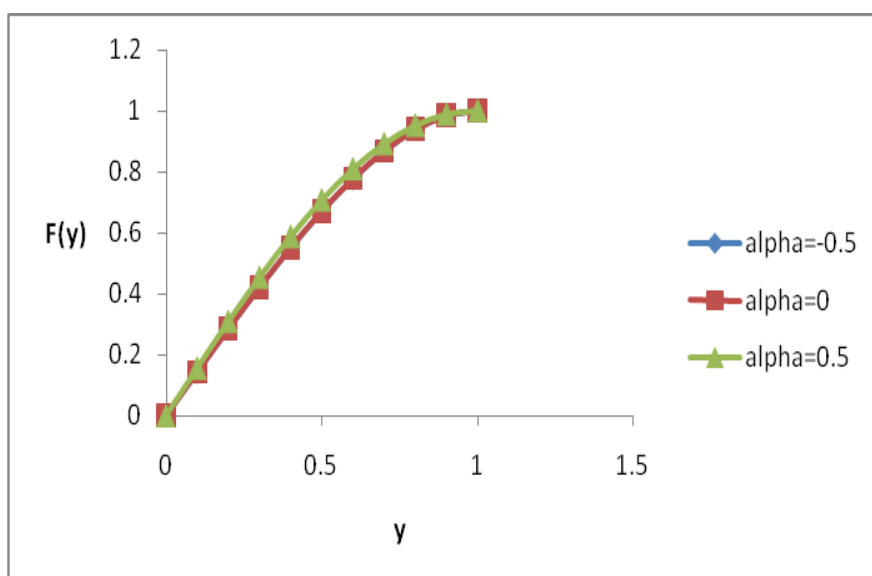
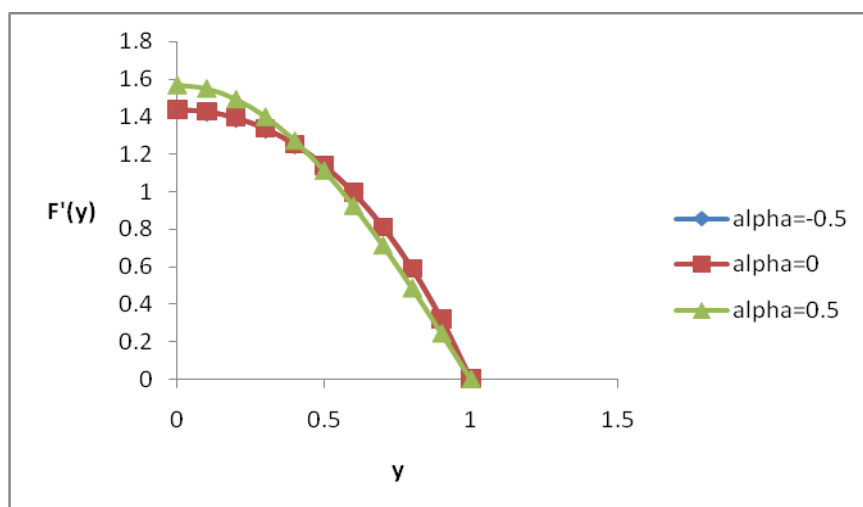


Figure 6 and 7: approximate solution of $F(y)$ and $F'(y)$ for different values of α when $Re=-1$

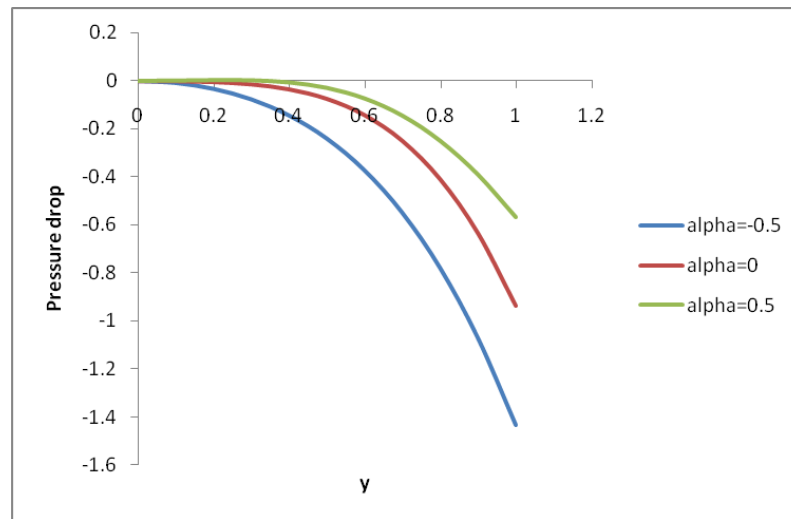


Figure7: The pressure drop in the normal direction changes shown over a range of α at $Re=-1$

Result and discussion:

- ✓ The objective of the present study was to apply QSC method to obtain an approximate solution of laminar flow in a rectangular domain bounded by two moving porous walls, which enable the fluid to enter or exit during successive expansion or contraction (Fig.1)
- ✓ (Fig.2,3,4) shows the comparison between HMA, SHAM, Numerical method and QSC method. It is verify that, it is acceptable agreement for proposed method for various values of permeation Reynolds number
- ✓ (Fig. 5,6) give a QSC solutions for the mean flow function $F(y) = -\frac{V}{C}$ and $F'(y) = \frac{\mu C}{x}$ for different
- ✓ permeable Reynolds number and non dimensional wall dilation rate
- ✓ For every level of injection or suction, in the case of expanding wall, increasing α leads to higher axial velocity near the center and lower axial velocity near the wall. The reason is that the flow toward the center becomes greater to make up for the space caused by the expansion of the wall and as a result, the axial velocity also becomes greater near the center.
- ✓ (Figure 7) shows that for every level of injection or suction, the absolute pressure changes in the normal direction is lowest near the central portion.

Conclusion

- ✓ In this study , the QSC method was successfully applied to find the approximate solution for laminar viscous flow in a rectangular domain bounded by two moving porous walls, which is enable the fluid to enter or exit during successive expansions or contractions. The result shows that For every level of injection or suction, in the case of expanding wall, increasing α leads to higher axial velocity near the center and lower axial velocity near the wall.

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