Scientific Journal of Impact Factor (SJIF): 4.72
e-ISSN (O): 2348-4470
p-ISSN (P): 2348-6406
International Journal of Advance Engineering and Research Development
Volume 4, Issue 06, June -2017

Extension of selected aspects offractional Brownian motion to set indexed framework

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Abstract: *The purpose of this article is to extend numerousaspectsof one-parameter fractional Brownian motion to set-indexed fractional Brownian motion. Most of the proofs are performed by "characterization of set-indexed fractional Brownian motion by flows". The characterization was proven by Herbin E. and Merzbach E. (see [HeMe3]), which saysthat a set-indexed process is a set-indexed fractional Brownian motion if and only if its projections on all the increasing paths are one-parameter time changed fractional Brownianmotions.*

Keywords:*Fractional Brownian motion, set indexed, flow, increasing path.*

Introduction

In this study, we extend the selected aspects of classical fractional Brownian motion to fractional Brownian motion, when the set index $\mathbf{A}$ is a compact set collection on a topologicalspace (T,τ). This fractional Brownian motion is called set-indexed fractional Brownian motion. The frame of a set-indexed fractional Brownian motion is not only a new step towards generalization of a classical fractional Brownian motion, but it provides a real tool in modeling.

Fractional Brownian motion finds applications in diverse fields such as finance, economics, biology, telecommunications, hydrology, physics and engineering. Whenever we are interested in processes with self-similarity, long memory, long-range dependence, Holder continuity, differentiability,stationary increments fractional Brownian motion is a natural candidate. Many authors have studied applications of fractional Brownian motion. For instance in mathematical finance, Cheridito [Ch] constructed arbitrage strategies using fractional Brownian motion, Comte and Renault [Co] presented continuous time models with long memory and with fractional Brownian motion, and Rogers [Ro] discussed use of fractional Brownian motion in modeling long range dependence of share returns. In engineering applications, for instance, in [Du] a queue with an infinite buffer space and fractional Brownian motion as a long-range dependent input have been studied. In [No],Norros presented a model for connectionless traffic using fractional Brownian motion. In [Le], the self-similarity of fractional Brownian motion has been studied in capturing fractal behavior in Ethernet local area network traffic.

Herbin E. and Merzbach E. proved the "characterization of set-indexed fractional Brownian motion by flows", (which saysthat a set-indexed process is a set-indexed fractional Brownian motion if and only if its projections on all the increasing paths are one-parameter time changed fractional Brownian motions (see [HeMe1])) is the key to most of the proofs in this article. It is of great importance since it allows us to "divide and conquer". Therefore, numerousproofs of set-indexed fractional Brownian motion can be recovered, by reducing to one-dimensional fractional Brownian motion.We extend some selected aspectsto theset indexed fractional Brownian motion for the following issues:self-similarity, stationary increment, scaling, α -Holder continuous of order $0<\alpha<H$, not Holder continuous of order $\alpha = H$, non-differentiability etc.

Preliminaries

The set-indexed framework:

Let (T,τ) denote a non-void σ -compact connected topological space. In set indexed works (see [MeYo], [HeMe1], [IvMe]) , processes and filtrations will be indexed by a nonempty class $\mathbf{A}$ of compact connected subsets of T is called an indexed collection if it satisfies the following:

1. $\varnothing \in \mathbf{A}$. In addition, there is an increasing sequence (B_n) of sets in $\mathbf{A}$ such that $T=\bigcup_{n=1}^{\infty} B_n^{\circ}$.
2. $\mathbf{A}$ is closed under arbitrary intersections and if $A,B\in\mathbf{A}$ are nonempty, then $A\cap B$ is nonempty. If (A_i) is an increasing sequence in $\mathbf{A}$ and if there exists n such that $A_i \subseteq B_n$ for every i, then $\overline{\bigcup_i A_i} \in \mathbf{A}$.
3. $\sigma(\mathbf{A}) = \mathbf{B}$ where $\mathbf{B}$ is the collection of Borel sets of T.
4. There exist an increasing sequence of finite sub-classes $\mathbf{A}_n = \{A_1^n,\ldots,A_{k_n}^n\} \subseteq \mathbf{A}$ closed under intersection with $\varnothing, B_n \in \mathbf{A}_n(\mathbf{u})$ ($\mathbf{A}_n(\mathbf{u})$ is the class of union of sets in $\mathbf{A}_n$), and a sequence of functions $g_n : \mathbf{A} \to \mathbf{A}_n(\mathbf{u}) \cup T$ such that:

i. g_n preserves arbitrary intersections and finite unions.

ii. For each $A \in \mathbf{A}$, $A \subseteq g_n(A)^{\circ}$ and $A = \cap_n g_n(A)$, $g_n(A) \subseteq g_m(A)$ if $n \geq m$

iii. $g_n(A) \cap A' \in \mathbf{A}$ if $A, A' \in \mathbf{A}$ and $g_n(A) \cap A' \in \mathbf{A}_n$ if $A \in \mathbf{A}$ and $A' \in \mathbf{A}_n$.

iv. $g_n(\varnothing) = \varnothing$ for all n.

(Note: $\overline{(\cdot)}$ and $(\cdot)^{\circ}$ denote respectively the closure and the interior of a set).

Examples of topological spaces T and indexed collections $\mathbf{A}$:

a. The classical example is $T = \Re_+^d$ and $\mathbf{A} = \mathbf{A}(\Re_+^d) = \{[0,x] : x \in \Re_+^d\}$.

b. The example (a) may be generalized as follows. Let $T = \Re_+^d$ and take $\mathbf{A}$ to be the class of compact *lower sets*, i.e. the class of compact subsets A of T satisfying $t \in A$ implies $[0,t] \subseteq \mathbf{A}$ (We denote the class of compact *lower sets* by $\mathbf{A}(Ls)$).

We will require other classes of sets generated by $\mathbf{A}$. The first is $\mathbf{A(u)}$, which is the class of finite unions of sets in $\mathbf{A}$. We note that $\mathbf{A(u)}$ is itself a lattice with the partial order induced by set inclusion. Let $\mathbf{C}$ consists of all the subsets of T of the form

$$C = A \setminus B, A \in \mathbf{A}, B \in \mathbf{A(u)}.$$

In addition, let A^{ss} be any finite sub-semilattice of $\mathbf{A}$ closed under intersection. For $A \in A^{ss}$, define the left neighborhood of A in A^{ss} to be a set $C_A = A \setminus \bigcup_{B \in A^{ss}, B \subset A} B$. We note that $\bigcup_{A \in A^{ss}} A = \bigcup_{A \in A^{ss}} C_A$ and that the latter union is disjoint. The sets in A^{ss} can always be numbered in the following way: $A_0 = \varnothing'$, ($\varnothing' = \bigcap_{A \in \mathbf{A}, A \neq \varnothing} A$, note that $\varnothing' \neq \varnothing$) and given $A_0, \ldots, A_{i-1}$, choose A_i to be any set in A^{ss} such that $A \subset A_i$ implies that $A = A_j$, some $j = 1, \ldots, i-1$. Any such numbering $A^{ss} = \{A_0, \ldots, A_k\}$ will be called "consistent with the strong past" (i.e., if C_i is the left-neighborhood of A_i in A^{ss}, then $C_i = \bigcup_{j=0}^{i} A_j \setminus \bigcup_{j=0}^{i-1} A_j$ and $C_i \bigcap A_j = \varnothing$, for all $j = 0, \ldots, i-1$, $i = 1, \ldots, k$).

A set-indexed stochastic process $X = \{X_A : A \in \mathbf{A}\}$ is additive if ithas an (almost sure) additive extension to $\mathbf{C}$: $X_{\varnothing} = 0$ and if $C, C_1, C_2 \in \mathbf{C}$ with $C = C_1 \bigcup C_2$ and $C_1 \bigcap C_2 = \varnothing$ then almost surely $X_C = X_{C_1} + X_{C_2}$. In particular, if $C \in \mathbf{C}$ and $C = A \setminus \bigcup_{i=1}^{n} A_i$, $A, A_1, \ldots, A_n \in \mathbf{A}$ then almost surely

$$X_C = X_A - \sum_{i=1}^{n} X_{A \cap A_i} + \sum_{i<j} X_{A \cap A_i \cap A_j} - \ldots + (-1)^n X_{A \cap \bigcap_{i=1}^{n} A_i}.$$

Set indexed fractional Brownian motions:

The key ingredient for characterization of set-indexed fractional Brownian motion is the use of a flow; that is, increasing function from interval $[a,b]$ to $\mathbf{A(u)}$. The judicious construction of a flow with particular properties permits us to reduce the general problem to one dimension. The definition of a flow is as follows:

Let $[a,b] \subset \Re_+$. A strict flow (shortly, flow) is defined to be a continuous increasing function $f : [a,b] \to \mathbf{A(u)}$, i.e. such that

a. $\forall s,t \in [a,b];\ s < t \Rightarrow f(s) \subset f(t)$

b. $\forall s,t \in [a,b);\ f(s) = \bigcap_{v>s} f(v)$

c. $\forall s,t \in (a,b);\ f(s) = \overline{\bigcup_{u<s} f(u)}$.

The notion of flow was introduced in [CaWa] and used by several authors[Da], [He].

Given a set indexed stochastic process X and the flow $f : [a,b] \to \mathbf{A(u)}$, we define a process Y indexed by $[a,b]$ as follows: $Y_s = X_{f(s)} = X_s^f$ for all $s \in [a,b]$.

Definition 1. A positive measure μ on $(T,\mathbf{B})$ is called strictly monotone on $\mathbf{A}$ if: $\sigma_{\varnothing'}=0$ and $\sigma_A<\sigma_B$ for all $A\subset B$, $A,B\in\mathbf{A}$. The collection of these measures is denoted by $M(\mathbf{A})$.

Recall that the fractional Brownian motion (fBM) $X^H=\{X_t^H:t\geq 0\}$ is defined to be a mean-zero Gaussian process with the covariance function

$$E[X_s^H X_t^H]=\tfrac{1}{2}\left[s^{2H}+t^{2H}-|t-s|^{2H}\right] \text{ for all } s,t\geq 0$$

(Equivalently, $E[(X_s^H-X_t^H)^2]=|t-s|^{2H}$ for all $s,t\geq 0$). This process has a parameter $H\in(0,1)$, called the Hurst parameter or the Hurst index. The natural set-indexed extension of this process is to substitute the term $|t-s|^{2H}$ with $\mu(A\Delta B)^{2H}$:

Definition 2. Let $\mu\in M(\mathbf{A})$. We say that $X^H=\{X_A^H:A\in\mathbf{A}\}$ is a set indexed fractional Brownian motion of parameter H (sifBM) if X^H is a centered Gaussian processsuch that $E[X_A^H X_B^H]=\tfrac{1}{2}[\mu(A)^{2H}+\mu(B)^{2H}-\mu(A\Delta B)^{2H}]$ for all $A,B\in\mathbf{A}$, where $0<H\leq\tfrac{1}{2}$.

(For more details about sifBMsee[HeMe1],[HeMe2], [HeMe3])

Lemma 1: Let $A^{SS}=\{\varnothing'=A_0,\ldots,A_k\}$ be any finite sub-semilattice of $\mathbf{A}$ equipped with a numbering consistent with the strong past.

Then there exists a flow $f:[0,k]\rightarrow\mathbf{A}(\mathbf{u})$ such that the following are satisfied:

1. $f(0)=\varnothing'$, $f(k)=\bigcup_{j=0}^{k}A_j$
2. Each left-neighbourhood C generated by A^{SS} is of the form $C=f(i)\setminus f(i-1)$ for all $1\leq i\leq k$.
3. If $C=f(t)\setminus f(s)$ then $C\in\mathbf{C}(\mathbf{u})$ and $F_{f(s)}\in\mathbf{G}_C^*$.

The proof appears in [IvMe].

Definition 3.A set-indexed process $X=\{X_A:A\in\mathbf{A}\}$ is said L^2-monotone outer-continuous if X_A is:

a. Square-integrable for all $A\in\mathbf{A}$.
b. For any decreasing sequence $\{A_n\}_{n=1}^{\infty}\in\mathbf{A}$, $\lim_{n\to\infty}E[|X_{A_n}-X_{\cap_{m=1}^{\infty}A_m}|^2]=0$.

Theorem 1 is the key to most of the proofs in this article. Therefore, many of proofs of sifBMcan be recovered, by reducing to one-dimensional fBM. It is the important “bridge” from sifBM to fBM and from that we extend following issues: self-similarity, stationary increment, scaling, α-Holder continuous of order $0<\alpha<H$, not α-Holder continuous of order $\alpha=H$, non-differentiabilityetc.

Theorem 1(The characterization of set-indexedfractional Brownian motion by flows): Let $X=\{X_A:A\in\mathbf{A}\}$ be a L^2-monotone outer-continuousset-indexed stochastic process and $\mu\in M(\mathbf{A})$. X issifBM if and only if the process X^f is time-change fractionalBrownian motion of parameter $0<H<\tfrac{1}{2}$ for all flows $f:[a,b]\rightarrow\mathbf{A}(\mathbf{u})$. (The process X^f is called a time-change fractional Brownian motion if there exists $\theta:[a,b]\rightarrow[a,b]$ such that $X^{f\circ\theta}$ is a fractional Brownian motion, for some a strict continuous flow $f:[a,b]\rightarrow\mathbf{A}(\mathbf{u})$), i.e. such that $E[(X_t^{f\circ\theta}-X_s^{f\circ\theta})^2]=|\mu(f(\theta(t)))-\mu(f(\theta(s)))|^{2H}$ for all $s,t\in[a,b]$).

The proof appears in [HeMe3], [HeMe1].

Remarks:

a. If $H=\tfrac{1}{2}$ then X^H is awell-known set indexed Brownian motion. ([HeMe3]).
b. The characterization of set-indexedBrownian motion by flows you can see in [MeYo].

From Theorem 1 and Lemma 1, we derive:

Lemma 2: Let $\mu\in M(\mathbf{A})$ and $X^H=\{X_A^H:A\in\mathbf{A}\}$ be a sifBM.

1. If $\{A_i\}_{i=1}^{k}$ be an increasing sequence in $\mathbf{A(u)}$ then there exists a flow $f:[0,k]\to\mathbf{A(u)}$, $f(0)=\varnothing'$ and $f(i)=A_i$ for all $1\le i\le k$, such that $\left(X^H\right)^f$ is a time-change fBM. (In other words, there exists $\theta:[a,b]\to[a,b]$ such that $\left(X^H\right)^{f\circ\theta}$ is afBM, for some a flow $f:[a,b]\to\mathbf{A(u)}$, i.e. such that $E\left[\left(\left(X^H\right)_t^{f\circ\theta}-\left(X^H\right)_s^{f\circ\theta}\right)^2\right]=|\mu(f(\theta(t)))-\mu(f(\theta(s)))|^{2H}$ for all $s,t\in[a,b]$.
2. If $\{A_i\}_{i=1}^{\infty}$ be an increasing sequence in $\mathbf{A(u)}$ then there exists a flow $f:[0,\infty)\to\mathbf{A(u)}$, $f(0)=\varnothing'$ and $f(i)=A_i$ for all $1\le i$, such that $\left(X^H\right)^f$ is a time-change fBM.

Proof.

a. Let $\{A_i\}_{i=1}^{k}$ be an increasing sequence in $\mathbf{A}$. Without loss of generality, we may assume that the sets $\{C_i\}_{i=1}^{k}$ are the left-neighborhoods of the sub-semilattice A^{ss} of $\mathbf{A}$ equipped with a numbering consistent with the strong past when $C_1=A_1$ and $C_i=A_i\setminus A_{i-1}$ for all $2\le i\le k$. According to Lemma 1, there exists a flow $f_k:[0,k]\to\mathbf{A(u)}$ such that each left-neighborhood generated by A^{ss} is of the form $C_i=f(i)\setminus f(i-1)$, $1\le i\le k$ and $F_{f_k(i)}\subseteq\mathbf{G}_C^*$. X is a sifBM then by Theorem 1, the process $\left(X^H\right)^{f_k}$ is a time-change fBM.

b. Notice that for each k, $f_k=f_{k+1}$ on $[0,k]$. Then, We can define the function $f:[0,\infty)\to\mathbf{A(u)}$ by $f(t)=f_{[t]+1}(t)$ for all t.

Continuity and differentiability

Definition 4. Let $\mu\in M(\mathbf{A})$ be a positive and continuous measure in $\mathbf{A}$. If $A\in\mathbf{A}$ and $\varepsilon\ge 0$ then define $D_A^{\varepsilon}=\{B\in\mathbf{A}:A\subseteq B,\mu(B\setminus A)=\varepsilon\}$. Denote by A^{ε} the element in D_A^{ε} and assume that $D_A^{\varepsilon}\ne\varnothing$. (Note: If $\varepsilon=0$ then $A^{\varepsilon}=A^0=A$)

Hereafter, we assume that the space T has a positive and continuous measure $\sigma\in M(\mathbf{A})$ in $\mathbf{A}$ such that for all $A\in\mathbf{A}$ there exists a A^{ε}, $\mu(A^{\varepsilon}\setminus A)=\varepsilon$ for all $\varepsilon\ge 0$.

The classical examples are: $T=\Re_+^d$ and $\mathbf{A}=\mathbf{A}(\Re_+^d)=\{[0,x]:x\in\Re_+^d\}$ or $\mathbf{A}=\mathbf{A}(Ls)$ when σ is Lebesgue measure (see [MeYo], [IvMe]).

Definition 5. Let $X=\{X_A:A\in\mathbf{A}\}$ be a set-indexed stochastic process.

a. X is said to be α-Holder continuous at $A\in\mathbf{A}$ if there exists $M>0,\delta>0$ such that $\left|X_{A^{\varepsilon}}-X_A\right|\le M\varepsilon^{\alpha}$ for all $0<\varepsilon<\delta$, for all $A^{\varepsilon}\in D_A^{\varepsilon}$ and $0<\alpha\le 1$.

b. X is said to be differentiable at $A\in\mathbf{A}$ if there exists a random variable Y such that $\lim_{\varepsilon\to 0^+}\frac{X_{A^{\varepsilon}}-X_A}{\varepsilon}-Y=0$ for all $A^{\varepsilon}\in D_A^{\varepsilon}$ and denote $Y=X'_A$ (the limit is mean in the sense of almost surely convergence).

Theorem 2. (α-Holder continuity and non-differentiability) Let $X^H=\{X_A^H:A\in\mathbf{A}\}$ be a sifBM.

a. If $0<\alpha<H$ then X_A^H is a α-Holder continuous at $A\in\mathbf{A}$, almost everywhere.

b. X is not differentiable at $A\in\mathbf{A}$, for almost all ω. (In other words, sifBM is nowhere differentiable almost surely), in fact, $\limsup_{\varepsilon\to 0^+}\left|\frac{X_{A^{\varepsilon}}^H-X_A^H}{\varepsilon}\right|=\infty$ with probability 1.

Proof.

Let $A \in \mathbf{A}$ and $A^{\varepsilon} \in D_A^{\varepsilon}$. According to Lemma 2, there exists a flow $f:[0,\infty) \to \mathbf{A}(\mathbf{u})$ and there exists $0 \le t \le t^{\varepsilon}$ such that $\left(X^H\right)^f$ is a time-change fBMand $A^{\varepsilon} = f(t^{\varepsilon}), A = f(t)$. Thus, there exists $\theta:[0,\infty) \to [0,\infty)$ and $0 \le \beta \le \beta^{\varepsilon}$ such that $\left(X^H\right)^{f\circ\theta}$ is a fBM and $A^{\varepsilon} = f(t^{\varepsilon}) = f(\theta(\beta^{\varepsilon})),\ A = f(t) = f(\theta(\beta))$

a. $\left(X^H\right)^{f\circ\theta}$ is a fBM (We recall that, if $W^H = \{W_t^H : t \ge 0\}$ is a fBMand $0 < \alpha < H$, then W is a α -Holder continuous path almost everywhere), then there exists a $M > 0, \delta > 0$ such that $\left|\left(X^H\right)^{f\circ\theta}_{\beta^{\varepsilon}} - \left(X^H\right)^{f\circ\theta}_{\beta}\right| \le M\varepsilon^{\alpha}$ for all $0 < \varepsilon < \delta$. But $\left|\left(X^H\right)^{f\circ\theta}_{\beta^{\varepsilon}} - \left(X^H\right)^{f\circ\theta}_{\beta}\right| = \left|X^H_{A^{\varepsilon}} - X^H_A\right|$ then there exists a $M > 0, \delta > 0$ such that $\left|X^H_{A^{\varepsilon}} - X^H_A\right| \le M\varepsilon^{\alpha}$ for all $0 < \varepsilon < \delta$, for all $A^{\varepsilon} \in D_A^{\varepsilon}$ and $0 < \alpha < H$.

b. $\left(X^H\right)^{f\circ\theta}$ is a fBM, (We recall that, if $W^H = \{W_t^H : t \ge 0\}$ is a fBM, then $\limsup_{\varepsilon \to 0^+} \left|\frac{W^H_{t+\varepsilon} - W^H_t}{\varepsilon}\right| = \infty$, with probability 1) then

$$\limsup_{\varepsilon \to 0^+} \left|\frac{X^H_{A^{\varepsilon}} - X^H_A}{\varepsilon}\right| = \limsup_{\varepsilon \to 0^+} \left|\frac{\left(X^H\right)^{f\circ\theta}_{\beta^{\varepsilon}} - \left(X^H\right)^{f\circ\theta}_{\beta}}{\varepsilon}\right| = \infty \text{ ,with probability 1.}$$

Moreover, X is not differentiable at $A \in \mathbf{A}$. □

The following theorem shows in particular that the sifBM is not α -Holder continuous of order $\alpha = H$.

Theorem 3: Let $X^H = \{X^H_A : A \in \mathbf{A}\}$ be a sifBM then for all $A \in \mathbf{A}$,

$$\limsup_{\varepsilon \to 0^+} \frac{X^H_{A^{\varepsilon}} - X^H_A}{\sqrt{2\varepsilon^{2H}\log\log(1/\varepsilon)}} = 1, \text{ almost surely.}$$

Proof.

Let $A \in \mathbf{A}$ and $A^{\varepsilon} \in D_A^{\varepsilon}$. According to Lemma 2, there exists a flow $f:[0,\infty) \to \mathbf{A}(\mathbf{u})$ and there exists $0 \le t \le t^{\varepsilon}$ such that $\left(X^H\right)^f$ is a time-change fBM and $A^{\varepsilon} = f(t^{\varepsilon}), A = f(t)$. Thus, there exists a $\theta:[0,\infty) \to [0,\infty)$ and $0 \le \beta \le \beta^{\varepsilon}$ such that $\left(X^H\right)^{f\circ\theta}$ is a fBMand $A^{\varepsilon} = f(t^{\varepsilon}) = f(\theta(\beta^{\varepsilon}))$, $A = f(t) = f(\theta(\beta))$. (We recall that, if $W = \{W_t^H : t \ge 0\}$ is a one-parameter fBM, then $\limsup_{\varepsilon \to 0^+} \frac{W^H_{t+\varepsilon} - W^H_t}{\sqrt{2\varepsilon^{2H}\log\log(1/\varepsilon)}} = 1$). Thus,

$$\limsup_{\varepsilon \to 0^+} \frac{X^H_{A^{\varepsilon}} - X^H_A}{\sqrt{2\varepsilon^{2H}\log\log(1/\varepsilon)}} = \limsup_{\varepsilon \to 0^+} \frac{\left(X^H\right)^{f\circ\theta}_{\beta^{\varepsilon}} - \left(X^H\right)^{f\circ\theta}_{\beta}}{\sqrt{2\varepsilon^{2H}\log\log(1/\varepsilon)}} = 1, \text{ almost surely. □}$$

Self-similarity

To study a set-indexed version of the notion of self-similarity for a set-indexed process, we need some assumptions about the set $\mathbf{A}$.

Definition 6. Let $\mu \in M(\mathbf{A})$ be a positive and continuous measure in $\mathbf{A}$. If $A \in \mathbf{A}$, $\mu(A) > 0$ and $\varepsilon > 0$ then

a. Define $\Pi_A^{\varepsilon} = \{B \in \mathbf{A} : \frac{\mu(B)}{\mu(A)} = \varepsilon\}$. Denote by $A^{[\varepsilon]}$ the element in Π_A^{ε} and assume that $\Pi_A^{\varepsilon} \ne \varnothing$.

b. Define $\Pi_{[A]}^{\varepsilon} = \{A^{[\varepsilon]} \in \Pi_A^{\varepsilon} : \forall B \in \mathbf{A}, \exists B^{[\varepsilon]} \in \Pi_B^{\varepsilon},\ (A \Delta B)^{[\varepsilon]} = A^{[\varepsilon]} \Delta B^{[\varepsilon]}\}$ .Denote by $[A]^{\varepsilon}$ the element in $\Pi_{[A]}^{\varepsilon}$ and assume that $\Pi_{[A]}^{\varepsilon} \ne \varnothing$.

Hereafter, we assume that the space T has a positive and continuous measure $\mu \in M(\mathbf{A})$ in $\mathbf{A}$ such that

a. For all $A \in \mathbf{A}$ there exists a $A^{[\varepsilon]}$, $\mu(A^{[\varepsilon]}) = \varepsilon\mu(A)$ for all $\varepsilon > 0$.

b. For all $A, B \in \mathbf{A}$ and for all $\varepsilon > 0$ there exists a $[A]^{\varepsilon} \in \Pi^{\varepsilon}_{[A]}$ and $[B]^{\varepsilon} \in \Pi^{\varepsilon}_{[B]}$ such that $[A\Delta B]^{\varepsilon} = [A]^{\varepsilon} \Delta [B]^{\varepsilon}$.

Example: $T = \Re^d_+$ and $\mathbf{A} = \mathbf{A}(\Re^d_+) = \{[0,x] : x \in \Re^d_+\}$. Let $A = [0,x]$, $B = [0,y]$ and $0 < \varepsilon$ where $x = (x_1, x_2, ..., x_d) \in \Re^d_+$ and $y = (y_1, y_2, ..., y_d) \in \Re^d_+$ then there exists a $[A]^{\varepsilon} = [0, x_1\sqrt[d]{\varepsilon}] \times \cdots \times [0, x_d\sqrt[d]{\varepsilon}] \in \Pi^{\varepsilon}_A$ and $[B]^{\varepsilon} = [0, y_1\sqrt[d]{\varepsilon}] \times [0, y_2\sqrt[d]{\varepsilon}] \times \cdots \times [0, y_d\sqrt[d]{\varepsilon}] \in \Pi^{\varepsilon}_B$ such that $\mu([A]^{\varepsilon}) = \varepsilon\mu(A)$, $\mu([B]^{\varepsilon}) = \varepsilon\mu(B)$ and $[A\Delta B]^{\varepsilon} = [A]^{\varepsilon} \Delta [B]^{\varepsilon}$.

Definition 7. A set-indexed process $X = \{X_A : A \in \mathbf{A}\}$ is said to be self-similar of index H if $X_{[A]^{\varepsilon}} \overset{d}{=} \varepsilon^{2H} X_A$ for all $A \in \mathbf{A}$, for all $\varepsilon > 0$ and for all $[A]^{\varepsilon} \in \Pi^{\varepsilon}_{[A]}$ (the notation $\overset{d}{=}$ mean identical distribution).

Theorem 4(Self-similarity of sifBM): The set-indexed fractional Brownian motion $X^H = \{X^H_A : A \in \mathbf{A}\}$ is self-similar of index H.

Proof:For all $A, B \in \mathbf{A}$, we have

$$E[X^H_{[A]^{\varepsilon}} X^H_{[B]^{\varepsilon}}] = \tfrac{1}{2}[\mu([A]^{\varepsilon})^{2H} + \mu([B]^{\varepsilon})^{2H} - \mu([A\Delta B]^{\varepsilon})^{2H}] =$$

$$= \varepsilon^{2H} \cdot \tfrac{1}{2}[\varepsilon^{2H}\mu(A)^{2H} + \mu(B)^{2H} - \mu(A\Delta B])^{2H}] = \varepsilon^{2H} E[X^H_A X^H_B].$$

Therefore, the two mean-zero Gaussian processes $X_{[A]^{\varepsilon}}$ and $\varepsilon^{2H} X_A$ have the same law, for all $A \in \mathbf{A}$. □

Corollary: Let $X^H = \{X^H_A : A \in \mathbf{A}\}$ be a sifBM.

a. (Stationary increment)For any $\varepsilon > 0$, $A_0 \in \mathbf{A}$, the processes $W^H = \{W^H_A : A \in \mathbf{A}\}$ and X^H have the same probability distribution, when $W^H_A = \frac{1}{\varepsilon^H}\left(X^H_{[A]^{\varepsilon}} - X^H_{[A_0]^{\varepsilon}}\right)$.

b. (Scaling) For any $\varepsilon > 0$, the process $W^H = \{W^H_A : A \in \mathbf{A}\}$ is asifBM when $W^H_A = \frac{1}{\varepsilon^H} X^H_{[A]^{\varepsilon}}$

Proof.

a. For all $A, B \in \mathbf{A}$, we have

$$E\left[\left(W^H_A - W^H_B\right)^2\right] = \frac{1}{\varepsilon^{2H}} E\left[\left(X^H_{[A]^{\varepsilon}} - X^H_{[B]^{\varepsilon}}\right)^2\right] = \frac{1}{\varepsilon^{2H}} \mu([A\Delta B]^{\varepsilon})^{2H}$$

Based on Theorem 4, $E[W^H_A X^H_B] = \frac{1}{\varepsilon^{2H}} E[X^H_{[A]^{\varepsilon}} X^H_{[B]^{\varepsilon}}] = E[X^H_A X^H_B]$. Therefore, the two mean-zero Gaussian processes W^H_A and X_A have the same law, for all $A \in \mathbf{A}$.

b. According to Theorem 4, $E[W^H_A X^H_B] = \frac{1}{\varepsilon^{2H}} E[X^H_{[A]^{\varepsilon}} X^H_{[B]^{\varepsilon}}] = E[X^H_A X^H_B]$. □

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